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THE NUMBERS OF TRIPLE TANGENCIES OF SMOOTH SPACE CURVES

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§1. INTRODUCTION

LET $c: S^1 \to \mathbb{R}^3$, be a curve of class C^∞. For each affine plane P in $\mathbb{R}^3$, we define the total order of contact of P to c as the sum of the orders of contact of P to c at all contact points. For example if P is transverse to c, the total order of contact is equal to zero. If the plane P is tangent to c at only one point $c(t)$, and P doesn't osculate c at $c(t)$, then the total order of contact of P to c is equal to 1 (see Lemma 2.2). $A(2, 3)$ denote the set of all affine planes in $\mathbb{R}^3$, and A_r the subset of $A(2, 3)$ consisting of planes whose total order of contact to c are equal to r. Suppose c is in general position (we will define the precise meaning of this expression in Definition 2.6). A_r's with $r > 3$ are empty, and A_3 is a finite set. Each plane $P \in A_3$ is called a triple tangency of the curve c. As the integer 3 can be written in the three ways as the sums of positive integers, we have three types of triple tangencies. A triple tangency is called T-plane, C-plane or I-plane according as the number of tangent points is equal to 3, 2 or 1. In other words, $P \in A(2, 3)$ is a T-plane if and only if P has three tangent points to c which are not on a line, and P is not the osculating plane at any points. $Q \in A(2, 3)$ is a C-plane if and only if Q osculates c at a point $c(t)$ where the torsion $\tau(t)$ of c does not vanish, and Q has another tangent point $c(s)$, which is not on the tangent of c at $c(t)$, and at which Q does not osculate c. $R \in A(2, 3)$ is an I-plane if and only if R osculates c at a point where the torsion τ of c vanishes, and the first derivative τ' of τ does not vanish, and also R is not tangent to c at any other points, see Fig. 1.

In this paper we are interested in the relations among the numbers of T-planes, C-planes and I-planes. The first result in this direction is the following:

THEOREM 1. (Freedman[4]). *If a curve c in general position has no I-planes, then the number* $|T|(c)$ *of all T-planes of c is even.*

To prove this, he considered the dual variety c^* in $\mathbb{R}P^3 = A(2, 3) \cup \{\infty\}$ consisting of planes which are tangent to c. c^* is an immersed 2-torus with singular points. A T-plane corresponds to a triple point of c^*. If c has no I-plane, (in other words, if c has no stalls,) all the singular points of c^* are of cusp type. Hence c^* can be replaced by an immersed torus without singular points, leaving the number of triple points unchanged. Therefore Theorem 1 is a consequence of the fact that properly immersed 2-tori have even number of triple points.

Recently Theorem 1 was generalized as follows; for each I-plane R, the index $|\mathrm{Ind}|(R)$ is defined as the number of the transverse intersection points of c with R. Then we have the

THEOREM 2. (Banchoff *et al.*[5]). *If c is in general position, then* $|T|(c) \equiv |I|(c)/2$ (*mod* 2), *where* $|I|(c)$ = *the sum of* $|\mathrm{Ind}|(R)$ *for all I-planes of c.*

Note that $|\mathrm{Ind}|(R)$ is always even. The method of the proof is also based on the study

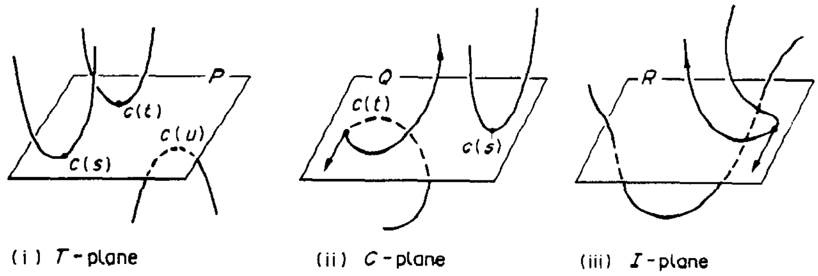


Fig. 1.

of geometry of c^*. Here c^* has, in general, swallowtails as well as cusps. The author received the manuscript[5] just after he had obtained the following result, which we are going to see.

We define the indices Ind(P) and Ind(Q) of T-planes P and C-planes Q by the integers ± 1, and indices Ind(R) of I-planes R by $\pm|\mathrm{Ind}|(R)$ (for the precise definitions, see Definitions 3.2, 3.3 and 3.4). We denote by $T(c)$, $C(c)$ and $I(c)$ the sums of all indices of the respective triple tangencies. Then we have the

THEOREM 3. *If c is in general position, we have* $2T(c) = C(c) = I(c)$.

Since $T(c) \equiv |T|(c) \pmod 2$ and $I(c)/2 \equiv |I|(c)/2 \pmod 2$, we have Theorem 2 as a corollary of Theorem 3. Theorem 3 is a consequence of the following:

THEOREM A. *If c is in general position, we have* $C(c) = I(c)$.
THEOREM B. *If c is in general position, we have* $6T(c) = 2C(c) + I(c)$.

The proofs of Theorems A and B are not based on the investigations of c^*, but on the idea used by Fabricius-Bjerre in [2] to prove the similar result about plane curves. The outline is the following; to a given curve c in general position, we will associate smooth families φ of planes in certain ways; $\varphi\colon J \to A(2, 3)$, where the parameter space J is homeomorphic to S^1. For each φ, we take a map $\tilde{\varphi}\colon J \tilde{\times} \mathbb{R}^2 \to \mathbb{R}^3$ such that the image $\tilde{\varphi}(\{t\} \tilde{\times} \mathbb{R}^2)$ is equal to $\varphi(t)$, and that $\tilde{\varphi}|_{\{t\} \tilde{\times} \mathbb{R}^2}$ is affine, where $J \tilde{\times} \mathbb{R}^2$ denotes the total space of $\mathbb{R}^2$-bundle over J induced by φ from the canonical $\mathbb{R}^2$-bundle over $A(2, 3)$, and $\{t\} \tilde{\times} \mathbb{R}^2$ denotes the fiber over $t \in J$. From the genericity of c, it follows that the pre-image $X = \tilde{\varphi}^{-1}(c(S^1))$ is a smooth submanifold of dimension 1 in $J \tilde{\times} \mathbb{R}^2$ (the actual definitions of X's are slightly modified). Let π denote the bundle projection of $J \tilde{\times} \mathbb{R}^2$. We will see that the critical points of $\pi|_X\colon X \to J$ and the boundary points of X correspond to triple tangencies. We will apply the Morse theory to this function to obtain relations among the numbers of critical points and boundary points.

Throughout this paper, curves are assumed to satisfy certain genericity conditions, which are the same as in [5]. The readers should refer[5] to see that the curves which satisfy these conditions make an open dense subset in the set of all C^∞-maps of S^1 into $\mathbb{R}^3$ with the Whitney C^∞-topology.

§2. PRELIMINARIES

We regard S^1 as the quotient space $\mathbb{R}/\mathbb{Z}$, and oriented canonically. We will use many S^1's at a time. Hence to distinguish them, we will write the name of variables as subscripts; S_t^1, $S_u^1, \ldots$. For three points t_1, t_2 and $t_3 \in S^1$, we use the inequalities $t_1 < t_2 < t_3$ to mean that these points are in the order of orientation of S^1. We denote by $\langle , \rangle$ the scalar product of $\mathbb{R}^3$.

Suppose that $c: S_t^1 \to \mathbb{R}^3$ is a C^∞-map, and that the unit tangent, the principal normal and the binormal are defined at each point $c(t)$. We denote these vectors by $e_1(t)$, $e_2(t)$ and $e_3(t)$, respectively. If the curve c is parameterized by arc length, e_1, e_2 and e_3 satisfy the Frenet–Serret formula:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix},$$

and κ and τ are called the curvature and the torsion of c. Suppose that c has non vanishing curvature at every point. We define maps ξ, ζ, $\tilde{\omega}$ and ψ as follows;

$$\begin{cases} \xi(t, \vartheta) = \cos(2\pi\vartheta) \cdot e_3(t) + \sin(2\pi\vartheta) \cdot e_2(t) \\ \zeta(t, \vartheta) = -\sin(2\pi\vartheta) \cdot e_3(t) + \cos(2\pi\vartheta) \cdot e_2(t) \\ \tilde{\omega}(t, \vartheta, x, y) = c(t) + x \cdot e_1(t) + y \cdot \zeta(t, \vartheta) \\ \psi(s, t, \vartheta) = \langle c(t+s) - c(t), \xi(t, \vartheta) \rangle \end{cases}$$

for $(s, t, \vartheta) \in S_s^1 \times S_t^1 \times S_\vartheta^1$ and $(x, y) \in \mathbb{R}^2$. The image $\tilde{\omega}(\{t\} \times \{0\} \times \mathbb{R}^2)$ is an affine plane which is called the osculating plane of c at $c(t)$. We denote this by $\omega(t)$; $\omega: S^1 \to A(2, 3)$, where $A(2, 3)$ denotes the set of all affine planes in $\mathbb{R}^3$.

LEMMA 2.1. *ψ satisfies the following relations*:

(1) $\psi(0, t, \vartheta) = \dfrac{\partial}{\partial s}\psi(0, t, \vartheta) = 0$ *for any* (t, ϑ).

(2) $\dfrac{\partial^2}{\partial s^2}\psi(0, t, \vartheta) = 0$ *if and only if* $\vartheta = 0$ *or* $1/2$.

(3) $\dfrac{\partial^3}{\partial s^3}\psi(0, t, 0)\begin{cases} > 0 & \text{if } \tau(t) > 0 \\ = 0 & \text{if } \tau(t) = 0 \\ < 0 & \text{if } \tau(t) < 0. \end{cases}$

(4) $\dfrac{\partial^3}{\partial s^2 \partial \vartheta}\psi(0, t, 0) > 0$ *for any* t.

(5) *Suppose* $\tau(t) = 0$.

$$\frac{\partial^4}{\partial s^4}\psi(0, t, 0)\begin{cases} > 0 & \text{if } \tau'(t) > 0 \\ = 0 & \text{if } \tau'(t) = 0 \\ < 0 & \text{if } \tau'(t) < 0. \end{cases}$$

Proof. These conditions don't depend on the choice of the parameter of c. They are directly deduced from the definition of ψ and the Frenet–Serret formula. Q.E.D.

Let $P \in A(2, 3)$ be tangent to c at c(t).

LEMMA 2.2. *The order n of contact of P to c at $c(t)$ is as follows*:

(1) $n = 0$ *if $e_1(t)$ is not parallel to P.*

(2) $n = 1$ *if* $e_1(t)$ *is parallel to* P *and* $e_2(t)$ *is not parallel to* P.
(3) $n = 2$ *if* $e_1(t)$ *and* $e_2(t)$ *are parallel to* P *and* $\tau(t) \neq 0$.
(4) $n = 3$ *if* $e_1(t)$ *and* $e_2(t)$ *are parallel to* P *and* $\tau(t) = 0$ *and* $\tau'(t) \neq 0$.
The proof is trivial.

Definition 2.3. The *total order of contact of* P *to* c is the sum of the order of contacts at every contact point of P to c.

Let φ: $(-\epsilon, \epsilon) \to A(2, 3)$ be a smooth curve in $A(2, 3)$, where ϵ is a small positive. A map $\tilde{\varphi}$: $(-\epsilon, \epsilon) \times \mathbb{R}^2 \to \mathbb{R}^3$ is called a *parameterization of* φ if $\tilde{\varphi}$ satisfies

(1) $\tilde{\varphi}|_{\{t\} \times \mathbb{R}^2}$ is an affine map for each fixed $t \in (-\epsilon, \epsilon)$,
(2) the image $\tilde{\varphi}(\{t\} \times \mathbb{R}^2)$ is equal to $\varphi(t)$.

Let π_t denote the projection of $\mathbb{R}^3$ to the orthogonal line $\varphi^{\perp}(t)$ of $\varphi(t)$ through the origin of $\mathbb{R}^3$. For each $t \in (-\epsilon, \epsilon)$, we define an affine subset $\operatorname{Ker} d^{\perp}\varphi(t)$ of $\varphi(t)$ as follows:

$$\operatorname{Ker} d^{\perp}\varphi(t) = \tilde{\varphi}\left(\{t\} \times \left(\text{the kernel of } \left(\pi_t \circ \frac{\partial \tilde{\varphi}}{\partial t}\right)\Big|_{\{t\} \times \mathbb{R}^2}\right)\right).$$

$\operatorname{Ker} d^{\perp}\varphi(t)$ doesn't depend on the choice of the parameterization $\tilde{\varphi}$ of φ.

Example 2.4. $\varphi = \omega$: $S^1 \to A(2, 3)$ be the osculating planes of c. Then $\operatorname{Ker} d^{\perp}\omega(t)$ contains the tangent of c at $c(t)$. If the torsion $\tau(t)$ is zero, then $\operatorname{Ker} d^{\perp}\omega(t) = \omega(t)$. In other words, the osculating plane rotates infinitesimally around the tangent, and if the torsion is zero at a point, then the osculating plane is infinitesimally stationary at the point.

Suppose that $\dim(\operatorname{Ker} d^{\perp}\varphi(t)) = 1$ for any $t \in (-\epsilon, \epsilon)$. Let $u(t)$ and $v(t)$ form a basis of $\varphi(t)$ and $u(t)$ be parallel to $\operatorname{Ker} d^{\perp}\varphi(t)$, and denote by $w(t) = u(t) \times v(t)$ the exterior product of these vectors. We say that φ *rotates positively* (resp. *negatively*) *with respect to the direction* $u(t)$ *of* $\operatorname{Ker} d^{\perp}\varphi(t)$, if $\langle w(t), v'(t) \rangle$ is positive (resp. negative). This definition does depend on the choice of the direction $u(t)$, but not on the choice of $v(t)$.

Example 2.5. The osculating plane $\omega(t)$ of c rotates positively (resp. negatively) with respect to $e_1(t)$ if the torsion $\tau(t)$ is positive (resp. negative).

In the following sections, we will suppose that the curves c satisfy
1. $c: S^1 \to \mathbb{R}^3$ are imbeddings.
2. The curvatures κ never vanish.
3. If $\tau(t) = 0$, then $\tau'(t) \neq 0$.
4. The total orders of contact of any planes to c are less than 4.
5. If $P \in A(2, 3)$ is tangent to c at three points, they are not on a line.
6. If an osculating plane $\omega(t)$ of c at $c(t)$ is tangent to c at $c(s)$ with $s \neq t$, then $c(s)$ is not on the tangent of c at $c(t)$.

Definition 2.6. If a curve $c: S^1 \to \mathbb{R}^3$ satisfies the above six conditions, we say that c is *in general position.*

The set of curves which are in general position is an open dense subset in the set of all C^∞*-maps of* S^1 *into* $\mathbb{R}^3$ *with the Whitney* C^∞*-topology*. For the proof of this fact, see [5].

§3. THE DEFINITIONS OF T, C AND I-PLANES AND THEIR INDICES

Throughout this section we suppose that $c: S^1 \to \mathbb{R}^3$ is in general position.

Definition 3.1. An affine plane is called a *triple tangency of c* if the total order of contact is equal to three. A triple tangency is called a *T-plane*, *C-plane* or *I-plane* according as the number of the tangent points is equal to 3, 2 or 1.

Let P be a T-plane of c, and $c(s)$, $c(t)$ and $c(u)$ be the tangent points with $s < t < u$. We denote by v the exterior product of $c(t) - c(s)$ and $c(u) - c(s)$. Since c is in general position v is not null, and also v is orthogonal to P. P induces the two closed half spaces, and H_v denotes the one which contains the vector $c(s) + v$;

$$H_v = \{x \in \mathbb{R}^3; \langle x - c(s), v \rangle \geqslant 0\}.$$

Definition 3.2. The *index* Ind (P) *of a T-plane P* is defined as follows:

$$\mathrm{Ind}(P) = \begin{cases} 1 & \text{if the number of the points of } s, t \text{ and } u \\ & \text{near which } c \text{ lies in } H_v \text{ is odd.} \\ -1 & \text{if this number is even.} \end{cases}$$

In Fig. 1 (i), $v = (c(t) - c(s)) \times (c(u) - c(s))$ looks up, and hence H_v is above P. c lies in H_v near $c(s)$ and $c(t)$. Hence Ind $(P) = -1$. We remark that we take the left hand system for the orientation of $\mathbb{R}^3$ to draw figures.

Let Q be a C-plane of c, and $c(s)$ and $c(t)$ be the tangent points with the orders of contact equal to 1 and 2, respectively. The unit tangent $e_1(t)$ and the principal normal $e_2(t)$ are parallel to Q. $e_1(s)$ is parallel to Q, but $e_2(s)$ is not parallel to Q.

Definition 3.3. The *index* Ind (Q) *of a C-plane Q* is defined as follows:

$$\mathrm{Ind}\,(Q) = \begin{cases} 1 \text{ if } \langle e_1(t) \times (c(s) - c(t)), e_2(s) \rangle > 0 \\ -1 \text{ if } \langle e_1(t) \times (c(s) - c(t)), e_2(s) \rangle < 0. \end{cases}$$

The C-plane Q in Fig. 1(ii) has the index Ind $(Q) = -1$.

Let the osculating plane $R = \omega(t_0)$ be an I-plane, that is, $\tau(t_0) = 0$ (and hence $\tau'(t_0) \neq 0$).

Definition 3.4. The *index* Ind (R) *of an I-plane R* is defined as follows:

$$\mathrm{Ind}\,(R) = \epsilon \text{ (the number of the transverse intersection points of } c \text{ and } R),$$

where $\epsilon = 1$ if $\tau'(t_0) < 0$, and $\epsilon = -1$ if $\tau'(t_0) > 0$.

The I-plane R in Fig. 1 (iii) has the index Ind $(R) = -2$. We should remark that if we reverse the orientation of the curve, all the indices will change the signs.

We denote by $T(c)$ $C(c)$ and $I(c)$ the sums of the indices of all respective triple tangencies:

$$T(c) = \sum_{T\text{-planes}} \mathrm{Ind}\,(P),\quad C(c) = \sum_{C\text{-planes}} \mathrm{Ind}\,(Q) \text{ and } I(c) = \sum_{I\text{-planes}} \mathrm{Ind}\,(R).$$

Now we state our theorems.

THEOREM A. *If c is in general position, we have* $C(c) = I(c)$.

THEOREM B. *If c is in general position, we have* $6T(c) = 2C(c) + I(c)$.

The actual calculations in the proofs of Theorems A and B come from the following fact; let X be homeomorphic to a finite disjoint union of S^1's, and closed intervals, and $f: X \to S^1$ be a smooth function with nondegenerate critical points in the interior and without critical points on the boundary. M and m denote the numbers of the relatively maximal and relatively minimal critical points of f, respectively, and N and n denote the numbers of the relatively maximal and relatively minimal boundary points, respectively. Then we have the

LEMMA 3.5. $M - m = (n - N)/2$.
The proof is trivial.

§4. THE PROOF OF THEOREM A

The maps ω, $\tilde{\omega}$, e_1, e_2, e_3, ψ, τ are as defined in §2. We suppose that the curve c is in general position, as before. We define the map $\tilde{\omega}_0$: $S^1 \times \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\tilde{\omega}(t, x, y) = \tilde{\omega}(t, 0, x, y) = c(t) + x \cdot e_1(t) + y \cdot e_2(t)$$

for $(t, x, y) \in S^1 \times \mathbb{R}^2$, and define the subset $X \subset S^1 \times \mathbb{R}^2$ by

$$X = (\tilde{\omega}\,|_{S^1 \times (\mathbb{R}^2 - \{0\})})^{-1}(c(S^1)).$$

$\tilde{\omega}_0$ is a parameterization of the osculating planes $\omega: S^1 \to A(2, 3)$. The genericity of c implies that X is a smooth submanifold in $S^1 \times \mathbb{R}^2$ of dimension 1. X is not necessarily closed in $S^1 \times \mathbb{R}^2$.

LEMMA 4.1. *The closure $\bar{X}$ of X in $S^1 \times \mathbb{R}^2$ is written as*

$$\bar{X} = X \cup \{(t, 0) \in S^1 \times \mathbb{R}^2; \tau(t) = 0\},$$

and $\bar{X}$ is a smooth submanifold without boundary in $S^1 \times \mathbb{R}^2$.

Proof. Since $c(S^1)$ is compact, X is proper in $S^1 \times (\mathbb{R}^2 - \{0\})$, and hence $\bar{X} - X$ is contained in $S^1 \times \{0\}$. Suppose the torsion $\tau(t_0) = 0$ at $c(t_0)$, (and hence $\tau'(t_0) \neq 0$). By Lemma 2.1, the germ of the function: $(s, t) \mapsto \psi(s, t + t_0, 0)$ at $(s, t) = (0, 0)$ is equivalent to the germ of the function: $(\bar{s}, \bar{t}) \mapsto \bar{s}^4 + \bar{t}\bar{s}^3$ at $(\bar{s}, \bar{t}) = (0, 0)$ as unfoldings of the functions of s and $\bar{s}$. The intersection of $c(S^1)$ with $\omega(t)$ is equal to the set $\{c(s); \psi(s, t, 0) = 0\}$. The zeroes of the function: $\bar{s} \mapsto \bar{s}^4 + \bar{t}\bar{s}^3$ are $\bar{s} = 0$ and $\bar{s} = -\bar{t}$. These facts imply that X is the graph of a smooth function of $t \in S^1$ near $(t_0, 0) \in S^1 \times \mathbb{R}^2$ except $t = t_0$, and this function of t can be trivially extended to a smooth function with t_0 as a point of the definition domain. Thus $\bar{X}$ is a smooth submanifold near $(t_0, 0) \in S^1 \times \mathbb{R}^2$. In the case that $\tau(t_0) \neq 0$, it is clear that $(t_0, 0) \notin \bar{X}$. Q.E.D.

Let π: $\bar{X} \to S^1$ denote the restriction on $\bar{X}$ of the projection of $S^1 \times \mathbb{R}^2$ onto the first factor. We see the relation between the critical points of π and C-planes in the

LEMMA 4.2. π has no critical points on $\bar{X} - X$. Each critical point $(t_0, x_0, y_0) \in \bar{X} \subset S^1 \times \mathbb{R}^2$ of π corresponds to a C-plane Q as follows; if $\tau(t_0) > 0$ (resp. $\tau(t_0) < 0$), then

$$\mathrm{Ind}\,(Q) = \begin{cases} 1 \text{ if } (t_0, x_0, y_0) \text{ is a relative minimum (resp. maximum)}, \\ -1 \text{ if } (t_0, x_0, y_0) \text{ is a relative maximum (resp. minimum)}. \end{cases}$$

Proof. From the proof of Lemma 4.1 and the genericity of c, it is clear that the critical points are not on $\pi^{-1}(t)$ with $\tau(t) = 0$. Let Q be a C-plane with the tangent points $c(t_0)$ and $c(s_0)$. Suppose $Q = \omega(t_0)$. We have $c(s_0) = \tilde{\omega}_0(t_0, x_0, y_0)$ for some $(x_0, y_0) \in \mathbb{R}^2$ with $y_0 \neq 0$. As we saw in Example 2.4, $\tilde{\omega}_0$ is regular at (t_0, x_0, y_0), since $\tau(t_0) \neq 0$ and $y_0 \neq 0$. We denote by v the orthogonal component of $(\partial/\partial t)\tilde{\omega}_0(t_0, x_0, y_0)$ to Q; $v = \langle (\partial/\partial t)\tilde{\omega}_0(t_0, x_0, y_0), e_3(t_0)\rangle \cdot e_3(t_0)$. We remark that $\langle v, e_2(s_0)\rangle$ is positive if and only if $(t_0, x_0, y_0) \in \bar{X}$ is a relative minimum of π. Suppose $\tau(t_0) > 0$ and $\mathrm{Ind}\,(Q) = 1$. By the definition of the index of C-planes we have

$$\begin{aligned} 0 &< \langle e_1(t_0) \times (c(s_0) - c(t_0)), e_2(s_0)\rangle \\ &= \langle e_1(t_0) \times (x_0 \cdot e_1(t_0) + y_0 \cdot e_2(t_0)), e_2(s_0)\rangle \\ &= y_0 \cdot \langle e_3(t_0), e_2(s_0)\rangle. \end{aligned}$$

From the definition of $\tilde{\omega}_0$ and this inequality, it follows that

$$\langle v, e_2(s_0)\rangle = \tau(t_0) \cdot y_0 \cdot \langle e_3(t_0), e_2(s_0)\rangle > 0.$$

Hence $\tau(t_0) > 0$ and $\mathrm{Ind}\,(Q) = 1$ imply that (t_0, x_0, y_0) is a relative minimum of π. For other cases the proofs are similar. Q.E.D.

Let $s_1, t_1, s_2, t_2, \ldots, s_a, t_a, s_{a+1} = s_1 \in S^1$ be such that $s_1 < t_1 < s_2 < t_2 < \cdots < s_a < t_a < s_1$, and that for $i = 1, 2, \ldots, a$

$$\begin{cases} \tau(t) > 0 \text{ if } s_i < t < t_i, \\ \tau(t) < 0 \text{ if } t_i < t < s_{i+1}. \end{cases}$$

Remark that $\{\omega(s_i), \omega(t_i); i = 1, \ldots, a\}$ is equal to the set of all I-planes. We define the subsets X_+ and X_- of $\bar{X}$ by

$$\begin{cases} X_+ = \cup_{i=1}^a X_{+,i} = \cup_{i=1}^a \pi^{-1}([s_i, t_i]) \\ X_- = \cup_{i=1}^a X_{-,i} = \cup_{i=1}^a \pi^{-1}([t_i, s_{i+1}]), \end{cases}$$

and denote by n_i^+ and n_i^- the numbers of intersection points of $c(S^1)$ with $\omega(s_i)$ and $\omega(t_i)$, respectively. In other words,

$$n_i^+ = -\mathrm{Ind}\,(\omega(s_i)) + 1, \text{ and } n_i^- = \mathrm{Ind}\,(\omega(t_i)) + 1. \tag{1}$$

For each critical point α of $\pi: X_{\pm,i} \to S^1$, we denote by $Q(\alpha)$ the C-plane which corresponds to α as we saw in Lemma 4.2. By Lemma 3.5 and Lemma 4.2, we have for each $i = 1, \ldots, a$,

$$\sum_{\alpha \in X_{+,i}} \mathrm{Ind}\,(Q(\alpha)) = (n_i^- - n_i^+)/2 \tag{2}$$

$$\sum_{\alpha \in X_{-,i}} \mathrm{Ind}\,(Q(\alpha)) = (n_{i+1}^+ - n_i^-)/2. \tag{3}$$

Since $\alpha \mapsto Q(\alpha)$ is a one to one correspondence between the set of the critical points of π and the set of the C-planes, the equality $C(c) = I(c)$ follows from (1), (2) and (3).

Q.E.D.

§5. THE PROOF OF THEOREM B

The curve c is again assumed to be in general position. $e_1(t)$, $e_2(t)$ and $e_3(t)$ denote the unit tangent, the principal normal and the binormal of c at $c(t)$, as before.

Definition 5.1. A plane $P \in A(2, 3)$ is called a *D-plane* of the curve c, if there exists a pair $(t_1, t_2) \in T^2 = S_t^1 \times S_t^1$ such that P is tangent to c at $c(t_1)$ and $c(t_2)$, and that $e_2(t_1)$ and $e_2(t_2)$ are not parallel to P. We call this pair a *D-pair*, and denote by $D \subset T^2$ the set of all D-pairs of c.

LEMMA 5.2. *Define two subsets D_1 and D_2 in T^2 as follows*:

$$D_1 = \{(t, t) \in T^2;\ \tau(t) = 0\}.$$

$$D_2 = \{(t_1, t_2) \in T^2;\ t_1 \neq t_2, \text{ and there exists a } C\text{-plane which is tangent to } c \text{ at } c(t_1) \text{ and } c(t_2)\}.$$

Then the closure $\bar{D}$ of D in T^2 is the union of D, D_1 and D_2. Moreover $\bar{D} \subset T^2$ is a smooth compact submanifold without boundary of dimension 1.

Briefly speaking, the closure of D is the union of D and $\{I\text{-planes and } C\text{-planes}\}$.

Proof. Let (t_1, t_2) be a D-pair. In case that $e_1(t_1)$ or $e_1(t_2)$ is not parallel to $c(t_1) - c(t_2)$, it is clear that in a small neighborhood of (t_1, t_2) in T^2, D is a smooth submanifold of dimension 1. Since c is in general position, it doesn't happen that both $e_1(t_1)$ and $e_1(t_2)$ are parallel to $c(t_1) - c(t_2)$. Hence D is a smooth submanifold of dimension 1 in T^2.

Now let (t_1, t_2) be a point in D_2. Then it follows that there exists a plane Q such that Q is tangent to c at $c(t_1)$ and $c(t_2)$, and that one of $e_2(t_1)$ and $e_2(t_2)$ is parallel to Q. Suppose $e_2(t_1)$ is parallel to Q. Then $e_2(t_2)$ is not parallel to Q, and $c(t_2)$ is not on the tangent of c at $c(t_1)$. Let $\tilde{\omega}$: $S_t^1 \times S_\vartheta^1 \times \mathbb{R}^2 \to \mathbb{R}^3$ denote the map defined in §2. We have $\tilde{\omega}(t_1, 0, x_0, y_0) = c(t_2)$ for some (x_0, y_0). For a fixed t sufficiently near t_1, the map: $(\vartheta, x, y) \mapsto \tilde{\omega}(t, \vartheta, x, y)$ is regular at $(0, x_0, y_0)$. Hence there exist two smooth maps $\vartheta(t)$ and $h_2(t)$ with $\vartheta(t_1) = 0$ and $h_2(t_1) = t_2$, defined on a small neighborhood of t_1, such that the plane $\{\tilde{\omega}(t, \vartheta(t), x, y);\ (x, y) \in \mathbb{R}^2\}$ is tangent to c at $c(h_2(t))$. Since the derivative $\vartheta'(t_1)$ is not null, and $\vartheta(t_1) = 0$, we have $\vartheta(t) \neq 0$ for $t \neq t_1$. This implies that the pair $(t, h_2(t))$ are D-pairs. Thus near (t_1, t_2), $\bar{D}$ is a smooth sub-manifold of dimension 1.

Next we take a pair (t_0, t_0) with $\tau(t_0) = 0$ (hence $\tau'(t_0) \neq 0$). For each unit vector $z \in S^2 \subset \mathbb{R}^3$, we define a smooth function f_z: $S^1 \to \mathbb{R}$ by $f_z(s) = \langle c(t_0 + s) - c(t_0), z\rangle$. By Lemma 2.1, we have at $(s, z) = (0, e_3(t_0))$

$$f_z = \frac{d}{ds} f_z = \frac{d^2}{ds^2} f_z = \frac{d^3}{ds^3} f_z = 0, \text{ and } \frac{d^4}{ds^4} f_z \neq 0.$$

Hence if we regard f_z as an unfolding of $f_{e_3(t_0)}$ with the parameter $z \in S^2$, $z \mapsto f_z$ is a versal unfolding, and equivalent to the unfolding $(u, v) \mapsto F_{(u,v)}$ defined by $F_{(u,v)}(w) = (w^4/4) + (uw^2/2) + vw$ for $u, v, w \in \mathbb{R}$. For each $a \in \mathbb{R}$, the function $F_{(-a^2, 0)}$ has three critical points $-a$, 0 and a, and $-a$ and a have the same critical value. This and the equivalence of the unfoldings imply that there exist two smooth functions:

$\mathbb{R} \ni a \mapsto s_i(a) \in S_s^1$ $(i = 1, 2)$ defined on a small neighborhood of $0 \in \mathbb{R}$ such that

(i) $s_1(0) = s_2(0) = 0$,

(ii) the derivatives $s_1'(0)$ and $s_2'(0)$ are not null, and have opposite signs,

(iii) there exists a certain $z = z(a) \in S^2$ smoothly dependent on $a \in \mathbb{R}$ with $z(0) = e_3(t_0)$ such that the function $f_{z(a)}$ has three critical points $s_1(a)$, 0, $s_2(a)$ in a certain neighborhood of 0 in S_s^1, and $s_1(a)$ and $s_2(a)$ have the same critical value.

For $a \neq 0$, the second derivatives of f_z at $s_1(a)$ and $s_2(a)$ are not null. Hence $(t_0 + s_1(a), t_0 + s_2(a))$ is a D-pair for each $a \in \mathbb{R}$ with $a \neq 0$. Therefore the pair (t_0, t_0) with $\tau(t_0) = 0$ is on $\bar{D}$, and near (t_0, t_0), $\bar{D}$ is a smooth submanifold.

Other pairs (t, t) with $\tau(t) \neq 0$ are not on $\bar{D}$ since the functions $G_u(w) = w^3 + uw$ for any fixed u cannot have a pair of critical points with the same critical value. This proves the lemma. Q.E.D.

For each D-pair (s, t), there exists uniquely a D-plane which is tangent to c at $c(s)$ and $c(t)$. This defines a function $q: D \to A(2, 3)$, and from the proof of Lemma 5.2, it follows that this function can be smoothly extended to $\bar{D}$. We denote this function also by $q: \bar{D} \to A(2, 3)$.

We distinguish two types of D-pairs.

Definition 5.3. Let $(t_1, t_2) \in D$, and $Q = q(t_1, t_2)$ the D-plane. If $c(t_1) + e_2(t_1)$ and $c(t_2) + e_2(t_2)$ lie on the same side of Q, then the D-pair (t_1, t_2) is called *exterior*, and if these vectors lie on opposite sides of Q, (t_1, t_2) is called *interior*. (See Fig. 2).
Let ϵ be a small positive number, and $h = (h_1, h_2): (-\epsilon, \epsilon) \to \bar{D} \subset T^2$ be a local coordinate of $\bar{D}$. Suppose $h(t) \in D$ for any $t \in (-\epsilon, \epsilon)$.

Definition 5.4. Suppose h satisfies the following:

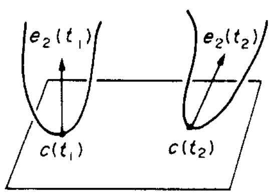
(N) $q \circ h: (-\epsilon, \epsilon) \to A(2, 3)$ rotates positively with respect to the direction $c(h_2(t)) - c(h_1(t))$ if $h(t)$ is exterior, or rotates negatively with respect to that direction if $h(t)$ is interior.

Then h is said to be *naturally oriented.*

Naturally oriented local coordinates define the orientation of D. This extends over $\bar{D}$ continuously as we see in the

LEMMA 5.5. *Naturally oriented local coordinates define a global orientation on each connected component of* $\bar{D}$.

Proof. Take a diffeomorphism $h = (h_1, h_2): S_u^1 \to$(a connected component of $\bar{D}$). We must investigate how to change the behaviors of $q \circ h(u)$ when $h(u)$ passes over a point of D_1 or D_2. First suppose $h(u_1) \in D_1$.



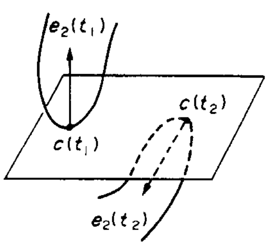


Fig. 2.

From the proof of Lemma 5.2, it follows that there exists an involution σ of S_u^1 such that $\sigma(u_1) = u_1$, and that $h_2(\sigma(u)) = h_1(u)$ and $h_1(\sigma(u)) = h_2(u)$ (and hence $q \circ h(\sigma(u)) = q \circ h(u)$) for any u sufficiently near u_1. We recall that $h(u)$ are exterior D-pairs for u sufficiently near u_1. If $q \circ h(u)$ rotates positively with respect to $c(h_2(u)) - c(h_1(u))$, $q \circ h \circ \sigma(u)$ rotates negatively with respect to this direction. Since $c(h_2(\sigma(u))) - c(h_1(\sigma(u))) = -(c(h_2(u)) - c(h_1(u)))$ and σ is orientation reversing, h satisfies the condition (N) just after u_1 if h satisfies it just before u_1.

Suppose $h(u_1) \in D_2$, and $q \circ h(u_1)$ osculates c at $c(h_2(u_1))$. If $h(u)$ is exterior for u just before u_1, $h(v)$ is interior for v just after u_1, and vice versa (since $\vartheta(h_2(u_1)) = 0$ and $\vartheta'(h_2(u_1)) \neq 0$, where ϑ is as defined in the proof of Lemma 5.2). Furthermore, $q \circ h$ changes the signs of rotation when the parameter u passes over u_1. Hence if h satisfies (N) just before u_1, h satisfies (N) just after u_1. Q.E.D.

For a moment, we fix a connected component $(\bar{D})_0$ of $\bar{D}$, and take a diffeomorphism $h = (h_1, h_2): S_u^1 \to (\bar{D})_0 \subset T^2 = S_t^1 \times S_t^1$ which is oriented with respect to the orientation (N) of $\bar{D}$. The composition map $\varphi = q \circ h: S_u^1 \to A(2, 3)$ induces an $\mathbb{R}^2$-bundle over S_u^1 from the canonical $\mathbb{R}^2$-bundle over $A(2, 3)$. We denote the total space of this bundle by $S_u^1 \tilde{\times} \mathbb{R}^2$. $\tilde{\varphi}: S_u^1 \tilde{\times} \mathbb{R}^2 \to \mathbb{R}^3$ denotes the parameterization of φ defined as the composition of obvious maps: $S_u^1 \tilde{\times} \mathbb{R}^2 \to A(2, 3) \times \mathbb{R}^3 \to \mathbb{R}^3$, and $\pi: S_u^1 \tilde{\times} \mathbb{R}^2 \to S_u^1$ denotes the projection. Using the component maps $h_i: S_u^1 \to S_t^1$ of $h = (h_1, h_2)$, we define the subset $S(u)$ of S_t^1 as follows; if $h_1(u) \neq h_2(u)$,

$$S(u) = \{t \in S_t^1;\ h_1(u) < t < h_2(u)\},$$

and if $h_1(u) = h_2(u)$, $S(u) = \emptyset$. The subset X in $S_u^1 \tilde{\times} \mathbb{R}^2$ is defined as the pre-image of $c(S(u))$'s by $\tilde{\varphi}: S_u^1 \tilde{\times} \mathbb{R}^2 \to \mathbb{R}^3$, more precisely

$$X = \bigcup_{u \in S_u^1} (\tilde{\varphi}|_{\pi^{-1}(u)})^{-1}(c(S(u)).$$

From the genericity of c, it follows that X is a smooth submanifold of dimension 1 in $S_u^1 \tilde{\times} \mathbb{R}^2$. The restriction of $\pi: S_u^1 \times \mathbb{R}^2 \to S_u^1$ on X is a smooth function of X. We will apply Lemma 3.5 to $\pi|_X$ to prove Theorem B. First we investigate the critical points of $\pi: X \to S_u^1$.

LEMMA 5.6. *Each critical point α of π corresponds to a T-plane $P(\alpha)$ in such a manner that*

$$\operatorname{Ind}(P(\alpha)) = \begin{cases} 1, & \text{if } \alpha \text{ is a relative maximum of } \pi, \\ -1, & \text{if } \alpha \text{ is a relative minimum of } \pi. \end{cases}$$

Proof. Let $u_0 = \pi(\alpha_0) \in S_u^1$ be fixed. We remark that α_0 is a critical point of π if and only if the plane $P(\alpha_0) = \varphi(u_0)$ is tangent to c at $\tilde{\varphi}(\alpha_0) \in \mathbb{R}^3$, as well as at $c(h_1(u_0))$ and $c(h_2(u_0))$, where $\tilde{\varphi}: S_u^1 \tilde{\times} \mathbb{R}^2 \to \mathbb{R}^3$ is the parameterization of φ given before. This means that $P(\alpha_0)$ is a T-plane. We write by $c(t_0)$ the third tangent point of $P(\alpha_0)$ to $c(S(u))$ $(t_0 \in S(u))$. Suppose $h(u_0) \in D$ is exterior. Then $\varphi: S_u^1 \to A(2, 3)$ rotates positively at $u = u_0$ with respect to $c(h_2(u_0)) - c(h_1(u_0))$. Let H_v denote the closed half space determined by $P(\alpha_0)$ and the exterior product $v = (c(t_0) - c(h_1(u_0))) \times (c(h_2(u_0)) - c(h_1(u_0)))$; $H_v = \{x \in \mathbb{R}^3;\ \langle x - c(h_1(u_0)), v\rangle \geqslant 0\}$. Remark that α_0 is a relative maximum if and only if in a small neighborhood of $c(t_0)$, the curve c lies in H_v (see Fig. 3. If c is as in the figure, α_0 is a relative minimum). On the other hand, c lies on the same side of $P(\alpha_0)$ in small neighborhoods of $c(h_i(u_0))$ $(i = 1, 2)$ since $h(u_0)$ is exterior. This implies that α_0 is a relative maximum if

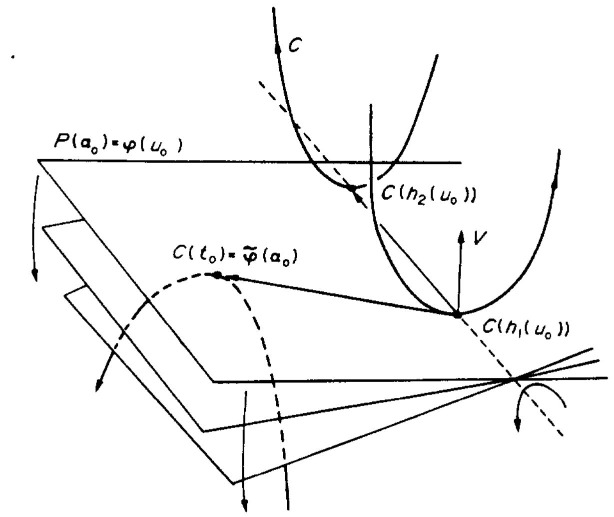


Fig. 3.

and only if Ind $(P(\alpha_0)) = 1$. This proves the lemma in the case that $h(u_0)$ is exterior. In the case that $h(u_0)$ is interior the proof is similar. Q.E.D.

Now we consider the boundary $\partial\tilde{X}$. Remark that $\partial\tilde{X}$ lies on the fibers $\pi^{-1}(u)$ of $S_u^1 \tilde{\times} \mathbb{R}^2$ such that $h(u) \in D_1 \cup D_2$, where D_1 and D_2 are defined in Lemma 5.2. In the following, we use the functions $\tilde{\omega}$, ϑ, f_z, $z(a)$ and $s_i(a)$ as used in the proof of Lemma 5.2.

Case 1. Let us first consider in a neighborhood of the fiber $\pi^{-1}(u_0)$ in $S_u^1 \tilde{\times} \mathbb{R}^2$ such that $h(u_0) \in D_1$. By definition, the torsion $\tau(t_0)$ with $t_0 = h_1(u_0) = h_2(u_0)$ is null. We recall that the osculating plane $R = \omega(t_0)$ of c at $c(t_0)$ is an I-plane, and that $|\text{Ind}\,(R)|$ is equal to the number of the transverse intersection points of R with c.

LEMMA 5.7. *The number of boundary points of $\tilde{X}$ on the fiber $\pi^{-1}(u_0)$ is equal to $|\text{Ind}\,(R)|$. These boundary points are relative minima of π if $\tau'(t_0) < 0$, or relative maxima if $\tau'(t_0) >$ 0.*

Proof. We saw, in the proof of Lemma 5.2, that the derivatives $h_1'(u_0)$ and $h_2'(u_0)$ have opposite signs. If $h_1'(u_0) < 0$, then for any $u < u_0$ sufficiently close to u_0, $c(S_t^1) \cap \varphi(u) = c(S(u)) \cap \varphi(u)$, and for any $u > u_0$ sufficiently close to u_0, $c(S(u)) \cap \varphi(u) \neq \emptyset$. Hence it suffices to show that $h_1'(u_0) < 0$ if and only if $\tau'(t_0) > 0$. We prove it only in the case $\tau'(t) > 0$. The proof of the other case will be similar.

Supposing $u_0 = 0$, we make the parameter $u \in S_u^1$ play the role of the parameter $a \in \mathbb{R}$ in the proof of Lemma 5.2. The unit vector $z(u)$ defined in (iii) in that proof is orthogonal to the D-plane $\varphi(u)$ with the D-pair $(h_1(u), h_2(u)) = (t_0 + s_1(u), t_0 + s_2(u))$. Since $\tau'(t_0) > 0$, we have $(d^4/ds^4)f_{z(0)}(0) > 0$ by Lemma 2.1 (5). Hence we have $(\partial/\partial u)((d^2/ds^2)f_{z(u)}(0)) > 0$ for $u < 0$. This implies that for $u < 0$

$$\left\langle e_2(t_0), \frac{\mathrm{d}}{\mathrm{d}u} z(u) \right\rangle > 0. \tag{1}$$

The D-pair $h(u)$ with $u \neq 0$ is exterior, and hence $\varphi(u)$ rotates positively with respect to $c(h_2(u)) - c(h_1(u))$. This is equivalent to that

$$\left\langle z(u) \times (c(h_2(u)) - c(h_1(u)), \frac{\mathrm{d}}{\mathrm{d}u} z(u) \right\rangle < 0. \tag{2}$$

Since $z(0) = e_3(t_0)$ and $\lim_{u \to 0} (c(h_2(u)) - c(h_1(u)))/u = (h_2'(0) - h_1'(0)) \times \|c'(t_0)\| \cdot e_1(t_0)$, the inequalities (1) and (2) imply $h_1'(0) < 0$. This completes the proof. Q.E.D.

Case 2. Next we consider in a neighborhood of the fiber $\pi^{-1}(u_0)$ in $S_u^1 \tilde{\times} \mathbb{R}^2$ such that $h(u)_0 = (h_2(u_0), h_1(u_0)) \in D_2$. Note that $\varphi(u_0)$ is a C-plane.

LEMMA 5.8. *$\bar{X}$ has a unique boundary point on $\pi^{-1}(u_0)$, which is a relative minimum of π if* $\mathrm{Ind}(\varphi(u_0)) = 1$, *or which is a relative maximum if* $\mathrm{Ind}(\varphi(u_0)) = -1$.

Proof. We prove the case $\mathrm{Ind}(\varphi(u_0)) = 1$ supposing that $\varphi(u_0)$ is the osculating plane of c at $c(h_2(u_0))$. In the other cases the proofs will be the same. Let $\psi: S_s^1 \times S_t^1 \times S_\vartheta^1 \to \mathbb{R}$ denote the function defined in §2. We consider the function: $(s, u) \mapsto \psi(s, h_2(u), \vartheta(u))$, where ϑ is as in the proof of Lemma 5.2. The assumption $\mathrm{Ind}(\varphi(u_0)) = 1$ implies that if $\tau(h_2(u_0)) > 0$ (resp. < 0)

$$\frac{\partial^3}{\partial s^2 \partial u} \psi(0, h_2(u_0), \vartheta(u_0)) > 0 \text{ (resp. } < 0)$$

$$\frac{\partial^3}{\partial s^3} \psi(0, h_2(u_0), \vartheta(u_0)) > 0 \text{ (resp. } < 0)$$

$$\frac{\mathrm{d}}{\mathrm{d}u} h_2(u_0) > 0.$$

For a fixed u sufficiently near u_0, the zeroes $\{s \in S_s^1;\ \psi(s, h_2(u), \vartheta(u)) = 0\}$ correspond to the intersection points of $\varphi(u)$ and $c(S_t^1)$. We consider the function: $(s, u) \mapsto \psi(s, h_2(u), \vartheta(u))$ as an unfolding of the function of s, and the graphs of these functions are in Fig. 4. For a number s with the small $|s|$, we have $h_2(u) + s \in S(u)$ if and only if $s < 0$. Hence on a small neighborhood of $c(h_2(u_0))$, the D-plane $\varphi(u)$ intersects with $c(S(u))$ if and only if $u > u_0$. This implies that $\bar{X}$ has one and only one relatively minimal boundary point on $\pi^{-1}(u_0)$. Q.E.D.

Now we know the correspondence between $\{$extrema of π: $\bar{X} \to S_u^1\}$ and $\{$triple tangencies$\}$.

Remark 5.9. (1) A T-plane P with $\mathrm{Ind}(P) = 1$ (resp. $\mathrm{Ind}(P) = -1$) contributes 3 times to the relative maxima (resp. minima) of π. (2) A C-plane Q with $\mathrm{Ind}(Q) = 1$ (resp. $\mathrm{Ind}(Q) = -1$) contributes twice to the boundary relative minima (resp. maxima) of

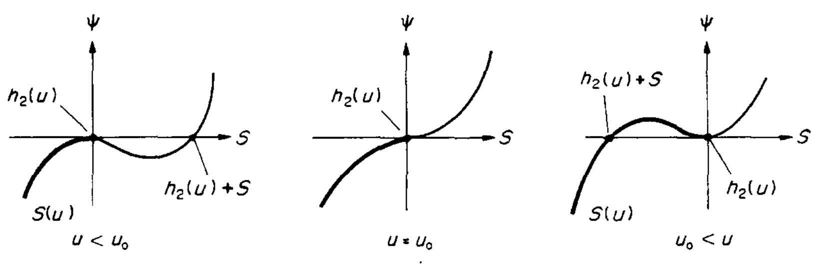


Fig. 4.

π. (3) An I-plane R with $\mathrm{Ind}(R) \geqslant 0$ (resp. $\mathrm{Ind}(R) \leqslant 0$) contributes $|\mathrm{Ind}(R)|$ times to the boundary relative minima (resp. maxima) of π.

Applying Lemma 3.5 to $\pi: \bar{X} \to S_u^1$, we finish the proof of Theorem B. Q.E.D.

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